Using dimensional analysis to estimate the yield of the *Trinity* nuclear test of July 16, 1945

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In 1941, the British physicist Sir Geoffrey Taylor was told by authorities *"that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission"* (Taylor 1950a). Based on his knowledge of conventional explosions, he published a report for the Ministry of Home Security attempting to determine what effects might be expected if such a nuclear explosion occurred. After the war the report was declassified and published (Taylor 1950a). Taylor[1] concluded that if a large amount of energy E was suddenly released in an infinitely concentrated form (a scenario not dissimilar to a nuclear explosion), then a spherical shock wave is propagated outwards whose radius R is related to the time t since the explosion by:

$$R = S(\gamma)\, t^{2/5}\, E^{1/5}\, \rho_0^{-1/5} \tag{1}$$

where ρ_0 is the ambient atmospheric density and the proportionality constant $S(\gamma)$ is a function of γ, the ratio of the specific heats of air (often called the adiabatic index). Recall that $\gamma = 1.4$ for diatomic gases (like air) under standard conditions. A schematic of the situation at some time t is depicted in Figure 1.

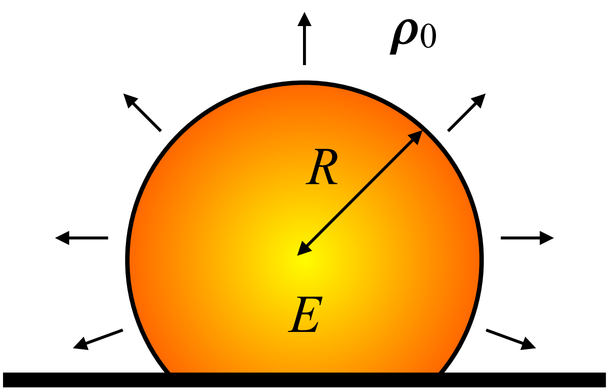

Figure 1: Cartoon of the *Trinity* explosion a short time after detonation. The fireball is expanding into the ambient air which has a density of ρ_0.

[1] Similar work was done independently around the same time by John von Neumann in the US and Leonid Sedov in the USSR.

It is obvious that R should depend positively on E and t (more energy = bigger bang), but it should also negatively depend on the ambient air density ρ_0 – an explosion pushing against denser air at sea level will not expand as quickly as one in less dense air at altitude or in a vacuum.

Although often (erroneously) described in the popular science literature as having performed a dimensional analysis to derive Equation 1, Taylor's approach was more rigorous and involved simultaneously solving the equation of motion, continuity equation and equation of state for an expanding spherical blast wave from a point source. Nonetheless, by assuming that the radius R depends *only* on E, t and ρ_0, we can arrive at the same general form as Taylor's equation using dimensional analysis. First, assume that R is proportional to E, t and ρ_0 to some mystery powers, i.e.

$$R \propto E^a \rho_0^b t^c \qquad (2)$$

where a, b and c are the exponents to be determined. We know that the dimensions of the four quantities are:

$$[R] = L \qquad [E] = ML^2T^{-2} \qquad [\rho_0] = ML^{-3} \qquad [t] = T$$

and so by considering the dimensions of mass, length and time in turn, we require both sides of Equation 2 to have the same exponent:

$$\begin{aligned} M &: 0 = a + b \\ L &: 1 = 2a - 3b \\ T &: 0 = -2a + c \end{aligned}$$

This set of equations can be easily solved by setting $a = -b$ from the first equation, using the second equation to find $b = -1/5$ and $a = 1/5$, and finally $c = 2/5$ from the third equation. Hence we have:

$$R = S \times E^{1/5} \rho_0^{-1/5} t^{2/5} \qquad (3)$$

for some (dimensionless) proportionality constant S, following Taylor's notation. Note that there is no reason why S needs to be one value for all radii and times. Given the extreme conditions inside the fireball it is quite conceivable that S could change depending on which processes dominate at a particular time.

One way to test the validity of this equation is to measure the radius of the fireball and plot it as a function of time. It is always easier to deal with straight lines when plotting, so we linearise Equation 3 by taking the (base-10) logarithm of both sides:

$$\log R = \frac{2}{5}\log t + \frac{1}{5}\log\frac{E}{\rho_0} + \log S.$$

Therefore, if we plot $\log R$ versus $\log t$ we expect the gradient to be exactly $\frac{2}{5}$ with a y-axis intercept c of:

$$c = \frac{1}{5}\log\frac{E}{\rho_0} + \log S = \frac{1}{5}\log\frac{E\,S^5}{\rho_0}. \qquad (4)$$

This is essentially the analysis Taylor performed in his second paper (Taylor 1950b) using declassified high-speed photos of the expanding *Trinity* fireball (Mack 1947). One such series of photos is shown in Figure 2. The length and time scales provided on the

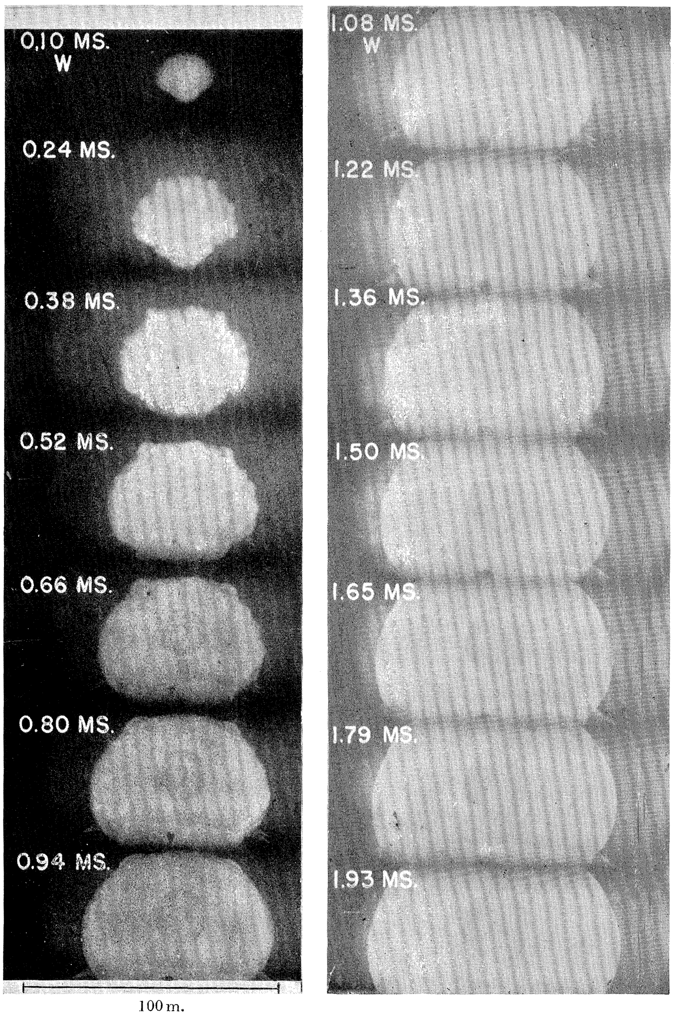

Figure 2: The expanding *Trinity* fireball captured in a series of photographs 0.1 ms to 1.93 ms after detonation. Note the length scale in the bottom-left corner. The explosion consumed the 30 m tower supporting the bomb in ∼0.6 ms.

frames allowed Taylor to accurately measure the radius of the fireball at each instant and produce a graph very similar to that shown in Figure 3[2].

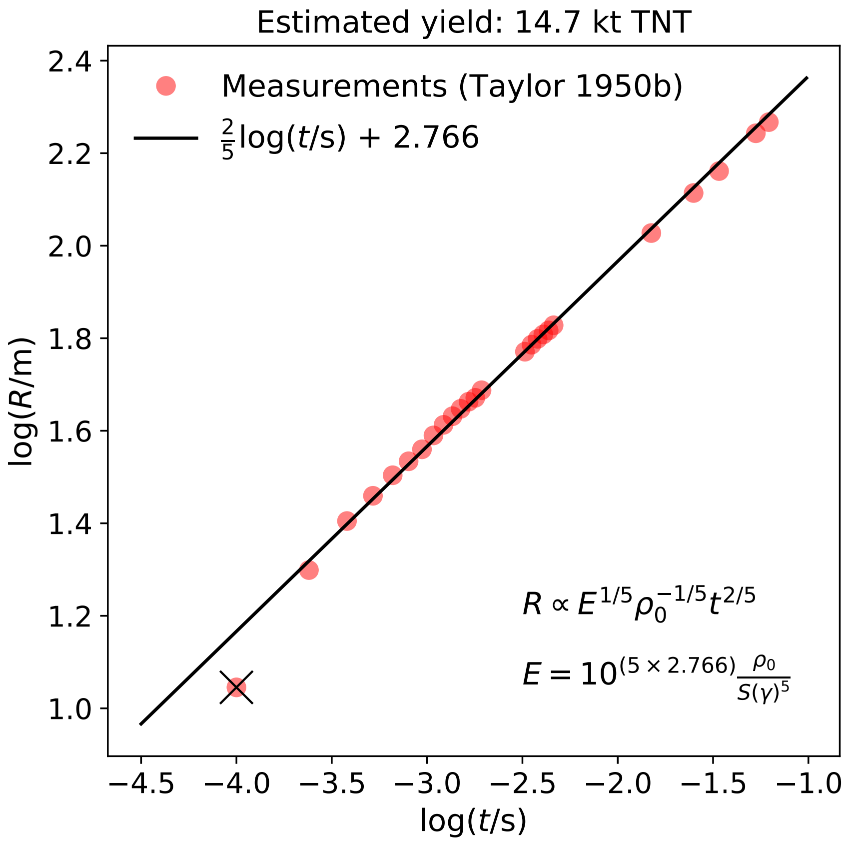

Figure 3: Plot of the *Trinity* fireball radius as a function of time, using measurements from Taylor (1950b). The solid line shows the $R \propto t^{2/5}$ relationship predicted by Taylor's analysis and our dimensional arguments. The y-intercept of the graph is related to the energy E and the air density ρ_0. The first data point was not included in the fit.

The solid line in Figure 3 was fitted with a fixed gradient of $\frac{2}{5}$ and a $\log R$ intercept of $c = 2.766$. The agreement between the predicted $\frac{2}{5}$ slope and the measurements is remarkable given the complex physics taking place inside the fireball during this time. Only the first data point (0.10 ms after detonation when the radius was 11 m) lies significantly off the linear trend, presumably due to the initial interaction of the rapidly expanding blast wave with the bomb casing and tower (also see Figure 2).

The excellent agreement between theory and measurement also implies that the explosion can be characterised by a single value of $S(\gamma)$ over a wide range of times. As Taylor notes in his second paper, *"this is surprising, because in those calculations it was assumed that air behaves as though γ, the ratio of the specific heats, is constant at all temperatures, an assumption which is certainly not true"*.

[2] Taylor actually plotted $\frac{5}{2}\log R$ versus $\log t$ to produce a graph with an expected gradient of 1.0, or an angle of 45° from either axis.

By rearranging Equation 4, we can easily calculate the energy of the explosion from the y-intercept c, air density ρ_0 and proportionality constant S:

$$E = 10^{(5\times c)}\frac{\rho_0}{S^5} \tag{5}$$

A pure dimensional analysis cannot provide a value for $S(\gamma)$ as it is dimensionless. The explosion energy E depends on S to the fifth power, so Taylor spent considerable effort determining its theoretical value. For dry air at standard temperatures, $\gamma = 1.4$ and he found $S(1.4) = 1.032$. Using this value and assuming an air density of 1.25 kg m^{-3}, Taylor estimated the *Trinity* test to have an explosion energy of $E = 71.4$ TJ, or equivalent to 16,800 tons (16.8 kt) of TNT[3]. Using an air density of 1.06 kg m^{-3} – more appropriate for the 1500 m altitude of the test site – we calculate a yield of:

$$E = 10^{(5\times 2.766)}\,\mathrm{m}^5\,\mathrm{s}^{-2}\,\frac{1.06\ \mathrm{kg\ m}^{-3}}{1.032^5} = \mathbf{61.5\ \ TJ}$$

or **14.7 kt TNT equivalent**. The official yield as determined by the US Department of Energy is 21 kt (DOE 2015), which includes the mechanical energy of the blast calculated here, as well as contributions from the light output and ionizing radiation. A re-analysis of Zirconium fission products published in 2016 estimated the yield at 22.1 ± 2.7 kt (Hanson et al. 2016). Despite containing very simplified physics, our yield agrees with these estimates within $\sim$30%.

Beware that in this example we were lucky that the proportionality constant S was so close to 1. This was not expected and will not be the case in all dimensional analyses, e.g. for a pendulum $T \propto \sqrt{\frac{L}{g}}$, with a proportionality constant of 2π. Had we assumed $S = 1$ exactly then the energy from Equation 5 would be 17.2 kt TNT, or 20.3 kt using the higher ρ_0 value – *coincidently* closer to the official yield.

Update 02/2021: Díaz (2021) applied Taylor's analysis to the August 2020 explosion of $\sim$2750 tons of ammonium nitrate in a warehouse in Beirut, Lebanon. After analysing several amateur videos from different perspectives to generate (R, t) measurements, they showed that the data follow the same $R \propto t^{2/5}$ relationship found by Taylor and estimated a yield of 0.6 ± 0.3 kt TNT equivalent. This is in good agreement with independent estimates from seismometers, infrasound data and the size of the crater. Their plot of fireball radius with time is reproduced in Figure 4 on the next page.

[3]Taylor used the English long ton (=1.0160 metric tons) so his value would be expressed today as 17.1 (metric) kt of TNT, where 1 kt TNT is defined to be exactly 4184 GJ.

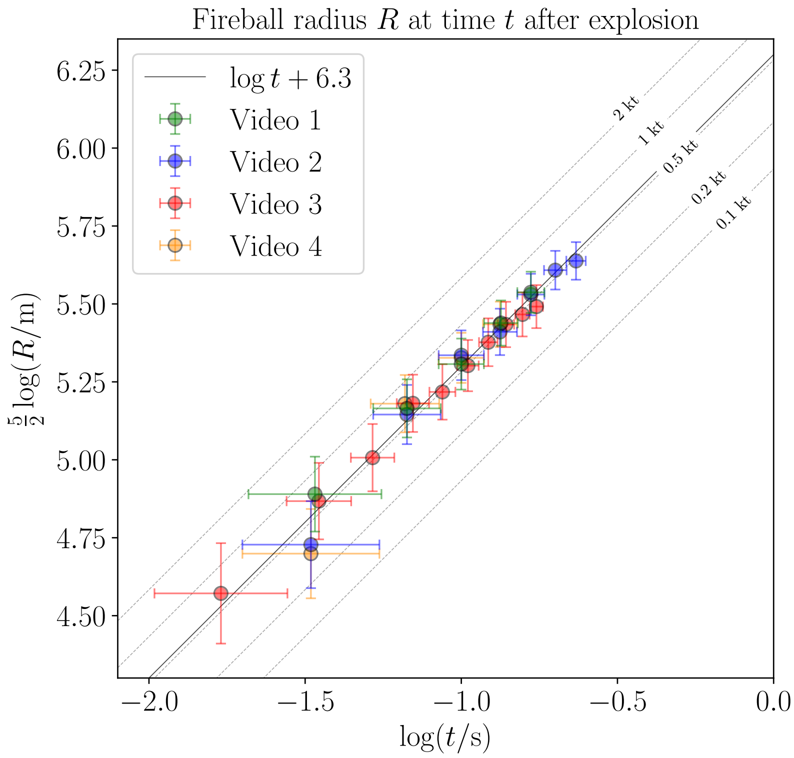

Figure 4: Plot from Díaz (2021) showing the progression of the Beirut fireball with time. Measurements were extracted from amateur videos at different vantage points to the explosion. The lines show kt TNT energies produced by a *hemispherical* blast wave. The data are consistent with an explosion energy of 0.6 $\pm$ 0.3 kt TNT.

References

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